Exercise 6

Show that if $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$, then

$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2,$$

where principal arguments are used.

Solution

Suppose that $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$. Then $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ lie in the first or fourth quadrants:

$$-\frac{\pi}{2} < \theta_1 + 2n_1\pi < \frac{\pi}{2} \tag{1}$$

and

$$-\frac{\pi}{2} < \theta_2 + 2n_2\pi < \frac{\pi}{2}, \tag{2}$$

where $n_1 = 0, \pm 1, \pm 2, \ldots$ and $n_2 = 0, \pm 1, \pm 2, \ldots$. We have

$$\arg z_1 = \arg r_1 e^{i\theta_1} = \theta_1 + 2n_1\pi.$$

Since this quantity on the right is between $-\pi/2$ and $\pi/2$, $\arg z_1$ can be replaced by the principal argument $\operatorname{Arg} z_1$. (The requirement to use $\operatorname{Arg} z$ is that $-\pi < \operatorname{Arg} z \leq \pi$.)

$$\arg z_1 = \operatorname{Arg} z_1$$

Similarly, we have

$$\arg z_2 = \arg r_2 e^{i\theta_2} = \theta_2 + 2n_2\pi.$$

Since this quantity on the right is between $-\pi/2$ and $\pi/2$, $\arg z_2$ can be replaced by the principal argument $\operatorname{Arg} z_2$.

$$\arg z_2 = \operatorname{Arg} z_2$$

Add each of the sides of inequalities (1) and (2).

$$-\pi < \theta_1 + \theta_2 + 2n_1\pi + 2n_2\pi < \pi \tag{3}$$

We also have

$$\begin{aligned}
\arg(z_1 z_2) &= \arg z_1 + \arg z_2 \\
&= \arg r_1 e^{i\theta_1} + \arg r_2 e^{i\theta_2} \\
&= (\theta_1 + 2n_1\pi) + (\theta_2 + 2n_2\pi) \\
&= \theta_1 + \theta_2 + 2n_1\pi + 2n_2\pi.
\end{aligned}$$

Since

$$-\pi < \arg(z_1 z_2) < \pi,$$

$\arg(z_1 z_2)$ can be replaced by the principal argument $\operatorname{Arg}(z_1 z_2)$. Therefore, because $\arg(z_1 z_2) = \arg z_1 + \arg z_2$,

$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2$$

for the case that z_1 and z_2 have positive real components.